**A20 FIND APPROXIMATE SOLUTIONS TO EQUATIONS NUMERICALLY USING ITERATION** **(higher tier)**

**ITERATION**

Iteration is the act of repeating a process, either to generate an unbounded sequence of outcomes, or with the aim of approaching a desired goal, target or result.

Each repetition of the process is also called an iteration, and the results of one iteration are used as the starting point for the next iteration.

Iteration is a way of solving equations. It is often used as a means of obtaining successively closer approximations to the solution of a problem. You would usually use iteration when you cannot solve the equation any other way.

**NOTATION**

 is the *n*th term of a sequence.

 is the (*n* +1)th term i.e. the term after the *n*th term.

 is the (*n* + 2)th term i.e. the second term after the *n*th term.

This is also the term after the (*n* + 1)th term.

 is the (*n* − 1)th term i.e. the term before the *n*th term.

 is often used as the starting point of an iterative solution

 is the first term of a sequence

**TYPES OF SEQUENCES**

Sequences may be defined by having enough terms to establish a pattern, for example, 1, 2, 4, 8, 16

Sequences may be defined by having a rule, for example, *n*th term = 4*n* + 3 producing 7, 11, 15, 19, 23

Sequences may be defined by giving the first term and the relationship between consecutive terms. The next example shows how this type of sequences works.

An **iterative sequence** is defined by giving the relationship between consecutive terms, e.g. The relationship, , is said to produce an iterative sequence.

**EXAMPLE 1**

Find the next five terms of the sequence given by where 

 so we can say that: It is useful to put the equation in words

any term = the term before it − 3 where *xn* + 1 = any term and *xn* is the term before it

 First term = 10; 2nd term = 1st term − 3

 3rd term = 2nd term − 3

 4th term = 3rd term − 3

 5th term = 4th term − 3

 6th term = 5th term − 3

Next five terms = 7, 4, 1, −2, −5

**EXERCISE 1:**

1. Write down the next five terms of the following sequences.

(a)  and 

(b)  and

(c)  and 

(d)  and 

(e)  and 

2. A school is expected to grow by 3% each year for the next five years.

The iteration  is used to work out the number of pupils each year and 

(a) What does  mean?

(b) Use the iteration to work out the number of pupils in five years’ time.

3. It has been predicted that the population of a small village is to decline at a rate of 2.5% each year for the next 8 years.

Use the iteration  to find the population in four years time of a village whose present population is 1848.

**LOCATING ROOTS**

Suppose the graph of *y* = f(*x*) is continuous and crosses the *x*-axis between *x* = *a* and *x* = *b*.

This situation can be seen in diagram 1 and in diagram 2.

**Diagram 1**

*a*

*y* = f(*x*)

*b*

f(*a*) < 0 and f(*b*) > 0

**Diagram 2**

*b*

*a*

*y* = f(*x*)

f(*a*) > 0 and f(*b*) < 0

In each case there is a **change in sign** for *y* either going from negative to positive (in Diagram 1) or from positive to negative positive (in Diagram 2).

If there is **a** **change in sign** for *y* for two particular values of *x* then we can say there is **a root** between these values of *x* and we can say that the equation f(*x*) = 0 will have a solution between these two values of *x*.

**EXAMPLE 2**

Show that the equation  has a solution between *x* = 1 and *x* = 2

As the equation is already equal to zero, we write it in function notation

 Write f(*x*) = given expression

Given *x* = 1 and *x* = 2

 Substitute for *x* = 1 in 

 Substitute for *x* = 2 in 

As f(1) > 0 i.e. positive and f(2) < 0 i.e. negative

there is a **change of sign**. You must write this statement

This shows that there is a solution between *x* = 1 and *x* = 2

**EXAMPLE 3**

Show that the equation  has a root between *x* = 2 and *x* = 3

We need to make the equation equal to zero,



Write the equation in function notation

 Once f(*x*) = 0, write f(*x*) = given expression

Given *x* = 2 and *x* = 3

 Substitute for *x* = 2 in 

 Substitute for *x* = 3 in 

As f(2) < 0 and f(3) > 0 there is a **change of sign**. You must write this statement

This shows that there is a root between *x* = 2 and *x* = 3

**NOTE:** We use the same method if asked for a **root** or a **solution** of the equation.

**EXERCISE 2:**

1. Show that the equation  has a solution between *x* = 2 and *x* = 3

2. Show that the equation  has a solution between *x* = 0 and *x* = 0.5

3. Show that the equation  has a solution between *x* = 1 and *x* = 2

4. Show that the equation  has a solution between *x* = 1 and *x* = 1.5

5\*. Does the equation  have a solution between *x* = −1 and *x* = 0.5 ?

**REARRANGING EQUATIONS**

In the exam you may be asked to rearrange an equation into an iteration formula. As before, basic algebraic rules apply.

Since these types of questions tend to be ‘show that …’, you must show all the intermediate steps to gain full marks.

**EXAMPLE 4**

Show that the equation  can be arranged to give 

As we can see that there is *x*2 and 1

in the 1st and 2nd equations Look at what you are trying to show

we start by getting rid of and see how it relates to what you are given

the −3*x* in the first equation.

 Start with the given equation

 Add 3*x* to each side

 Divide each term by 3

Hence,  Finish with a statement

Show that the equation  can be arranged to give 

**EXAMPLE 5**

We can see that 1 and *x*3 are on the right Look at what you are trying to show

and we need to divide by 5 and see how it relates to what you are given

 Start with the given equation

 Isolate the term 5*x*

 Divide each term by 5

Hence,  Finish with a statement

**EXERCISE 3:**

1. Show that the equation  can be rearranged to give 

2. Show that the equation  can be rearranged to give 

3. Show that the equation  can be rearranged to give 

4. Show that the equation  can be rearranged to give 

**EXAMPLE 6**

(a) Show that the equation  has a root between *x* = 4 and *x* = 5

(b) Show that the equation  can be arranged to give 

(c) Use the iteration , with , to find a solution to the equation correct to 1 decimal point.

1. Write the equation in function notation



As there is a root between *x* = 4 and *x* = 5

 Substitute for *x* = 4 in   Substitute for *x* = 5 in 

As f(4) < 0 and f(5) > 0

there is a **change of sign**. You must write this statement

so there is a root between *x* = 4 and *x* = 5

(b) We can see that 5*x* and 2 are on the right Look at what you are trying to show

and that we will need to square root and see how it relates to what you are given

 Start with the given equation

 add 5*x* and subtract 2 from each side

 square root both sides

(c) Given  and 

 1st term  where 

= 4.7958 = 4.8 to 1 d.p.

 2nd term  where 

= 4.6882 = 4.7 to 1 d.p.

 3rd term  where 

= 4.6304 = 4.6 to 1 d.p.

 4th term  where 

= 4.5992 = 4.6 to 1 d.p.

Hence, solution is **4.6** to 1 decimal place. We can stop as the rounding has settled to 4.6

**NOTE:** Always use your **unrounded** value when substituting.

**EXAMPLE 7**

(a) Show that the equation  can be arranged to give 

(b) Using  with , find the values of  and .

(c) Explain what the values of  and  represent.

(a)  We can see we want so divide each term by *x*2

 Divide each term by *x*2

 Add *x* to each side of the equation

Hence,  You must write this statement

1. Given  and 

 1st term  where 

 2nd term  where 

 3rd term  where 

(c) Each iteration is a better estimation of the solution of the equation 

**NOTE:** If you were asked to find a solution of the equation to 2 d.p. then the answer would be 3.28 as the value of *x* has settled to 3.28 to 2 d.p.

**EXERCISE 4**

**EXERCISE 4:**

1. (a) Show that the equation  can be rearranged to give 

This information can be used to obtain the iterative formula 

(b) Starting with  calculate the values of  and , giving all the figures on

your calculator display.

(c) Find one solution of  correct to 3 decimal places.

2. (a) Show that the equation  can be rearranged to give 

(b) Use the iteration , with , to find the values of  and .

3. (a) Show that the equation  has a solution between *x* = 0 and *x* = 1

(b) Show that the equation  can be arranged to give 

(c) Starting with  use the iteration formula  twice, to find an estimate

for the solution of 

4. An approximate solution to an equation is found using this iterative process

 and 

Work out the values of  and .

5. Show that the equation  can be rearranged to form the following iterative formulae.

(a) (i)  (ii) 

Use the following iterations where *x*0 = –2 in each case to find the values of *x*1, *x*2 and *x*3.

(b) (i)  (ii) 

6. (a) Show that the equation  has a solution between *x* = 4 and *x* = 5

(b) Show that the equation  can be arranged to give 

(c) Starting with  use the iteration formula  to find an estimate

for the solution of  to 2 decimal places.

7. (a) Complete the table for 

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *x* | 0 | 1 | 2 | 3 | 4 |
| *y* |  | −8 |  |  | 40 |

(b) Between which two consecutive integers is there a solution to the equation

? Give a reason for your answer.

(c) Show that the equation can be arranged to give 

(d) Starting with  use the iteration formula  to find an estimate

for the solution of  to 2 decimal places.

8. (a) show that  is a rearrangement of the equation 

(b) Use the iterative formula  together with a starting value of 

to obtain a solution of the equation  correct to 1 decimal place.

9. A computer uses the iteration  to find one solution for a quadratic equation.

(a) What quadratic equation is being solved?

(b) Find the positive solution of this equation.

10. (a) Show that  can be rearranged to give 

(b) Use the iterative formula  together with a starting value of 

to obtain a solution of the equation 

(c) The other solution is approximately 3.

What happens if we use 3 as the starting value in the iteration ?

(d) Show that  can be rearranged to give 

(e) Use the iterative formula  together with a starting value of 

to obtain the other solution of the equation 

**ANSWERS**

**Exercise 1**

1. (a) 5, 7, 9, 11, 13 (b) 5, 14, 41, 122, 365

(c) 0, −1, −3, −7, −15 (d) 4, 16, 256, 65536, 4294967296

(e) 

2. (a) Number of pupils at the start (b) 649

3. 1670

**Exercise 2**

1. f(2) = −6 ˂ 0 and f(3) = 13 ˃ 0

Hence, change of sign; equation will have a solution between these two values of *x*

2. f(0) = 2 ˃ 0 and f(0.5) = −0.875 ˂ 0

Hence, change of sign; equation will have a solution between these two values of *x*

3. f(1) = 1 ˃ 0 and f(2) = −3 ˂ 0

Hence, change of sign; equation will have a solution between these two values of *x*

4. f(1) = −1 ˂ 0 and f(1.5) = 0.875 ˃ 0

Hence, change of sign; equation will have a solution between these two values of *x*

5\*. f(−1) = −1 ˂ 0 and f(0.5) = 2 ˃ 0

Hence, change of sign; discuss with students why this equation will not have the a solution between these two values

**Exercise 3**

1. *x*2 + 4*x* – 6 = 0

4*x* = 6 – *x*2

*x* = 

*x* = 

2. *x*2 – 5*x* + 6 = 0

*x*2 = 5*x* – 6

*x* = 

3. *x*3 + *x* – 4 = 0

*x*3 = 4 – *x*

*x* = 

4. *x*3 + 3*x*2 – 5 = 0

*x*3 = 5 – 3*x*2

*x* = 

**Exercise 4**

1. (a) *x*2 – *x* = 0

*x*2 = *x* + 7

*x* = 

(b) *x*1 = 3.31662479

*x*2 = 3.211950309

*x*3 = 3.19561423

(c) 3.193

2. (a) *x*2 + *x* – 3 = 0

*x*(*x* + 1) = 3

*x* = 

(b) *x*1 = 1.5

*x*2 = 1.2

*x*3 = 1.364

3. (a) f(0) = −1 ˂ 0 and f(1) = 4 ˃ 0

Hence, change of sign

(b) *x*3 + 4*x* = 1

*x* = 1 – *x*3

*x* = 

(c) 0.246

4. *x*1 = −0.5

*x*2 = −0.391

*x*3 = −0.382

5. (a) (i) *x*3 – 5*x* + 7 = 0

5*x* = *x*3 + 7

*x*= (*x*3 + 7)

(ii) *x*3 – 5*x* + 7 = 0

*x*3 = 5*x* – 7

*x* = 

(b)(i) *x*1 = −0.2

*x*2 = 1.3984

*x*3 = 1.947

(ii) *x*1 = −2.571

*x*2 = −2.708

*x*3 = −2.739

6. (a) f(4) = −6 ˂ 0 and f(5) = 45 ˃ 0

Hence, change of sign

(b) *x*3 – 10*x* = 30

*x*3 = 30 + 10*x*

*x* = 

(c) 4.15

7. (a)

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| *x* | 0 | 1 | 2 | 3 | 4 |
| *y* | −4 | −8 | −6 | 8 | 40 |

(b) 2 and 3 due to the change of sign

(c) *x*3 – 5*x* – 4 = 0

*x*3 = 5*x* + 4

*x* = 

(d) 2.56

8. (a) *x* = 1 + 

*x* – 1 = 

(*x* – 1)(*x* – 3) = 11

*x*2 – *x* – 3*x* + 3 = 11

*x*2 – 4*x* – 8 = 0

(b) −1.5

9. (a) *x*2 + 3*x* – 2 = 0

(b) 0.56

10. (a) *x* = 

*x*(*x* – 1) = 5

*x*2 – *x* – 5 = 0

(b) −1.79

(c) Obtains the first solution from part (b)

(d) *x* = 

*x*(2*x* – 1) = *x*2 + 5

*x*2 – *x* – 5 = 0

(e) 2.79